1/ # Math 112 : Introductory Real Analysis

§ Lecture 6 (Feb 12, 2025)

Last time: metric spaces
open sets, closed sets, topology, limit points

Today: Compact sets

Def An open cover of a set E in a topological space X is a collection of open subsets of X such that $E \subseteq \bigcup_{\alpha} G_{\alpha}$.
$\{G_\alpha\}$

Def A subset K of a topological space X is said to be compact if every open cover of K contains a finite subcover.
That is, for every open cover $\{G_\alpha\}_{\alpha \in J}$ of K,
there is a finite set of indices $\{\alpha_1, \cdots, \alpha_n\} \subseteq J$ such that

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

Thm If F is a closed subset of a compact set K, then F is compact as well.

proof) If $\{G_\alpha\}$ is an open cover of F, then $\{G_\alpha\} \cup \{F^c\}$ is an open cover of K. Due to compactness of K, it has a finite subcover, and after removing F^c if necessary, we get a finite subcover for F as desired. ∎

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Thm If K is a compact set, then every infinite subset of K has a limit point in K.

proof) Let $E \subseteq K$ be an infinite subset, and suppose that no points in K were a limit point of E.

Then, for each $x \in K \setminus E$, there is an open neighborhood V_x of x disjoint from E, and for each $x \in E$ there is an open neighborhood V_x of x disjoint from $E \setminus \{x\}$.

The collection of these open sets $\{V_x\}_{x \in K}$ is an open cover of K but has no finite subcover, which contradicts compactness of K.

Therefore, E must have a limit point in K. ■

Thm If K is a subset of a metric space X such that every infinite subset of K has a limit point in K, then K is closed and bounded.

proof) If K weren't bounded, then for some fixed point $y \in K$, we can pick a sequence of points $\{x_n\}$ such that $d(x_n, y) > n$.

Then $\{x_n\}$ is an infinite subset of K without a limit point, which contradicts our assumption.

Hence K must be bounded.

(proof continued)

If K weren't closed, then there is some limit point $y \in \bar{K} \setminus K \subset X$.

Then, we can pick a sequence of points $\{x_n\}$ in K such that

$$d(x_n, y) < \frac{1}{n}.$$

Then, $\{x_n\}$ is an infinite subset of K, and it doesn't have any limit points in K,

since for any $z \in K$, $d(z, x_n) \geq d(z,y) - d(x_n, y) > d(z,y) - \frac{1}{n}$

$$\Rightarrow d(z, x_n) > \frac{d(z,y)}{2} \quad \text{for all } n > \frac{2}{d(z,y)}.$$

This contradicts our assumption, and hence K must be closed. ■

<u>Thm</u> (Heine-Borel theorem)

Let E be a subset of $\mathbb{R}^k$.

Then the followings are equivalent:

(a) E is compact

(b) Every infinite subset of E has a limit point in E

(c) E is closed and bounded.

proof) We've already shown that (a) $\Rightarrow$ (b) $\Rightarrow$ (c), so it remains to show that (c) implies (a).

4/ (proof continued)

Let $E \subset \mathbb{R}^k$ be closed and bounded.

Since it is bounded, it is contained in some closed k-dimensional cube $I \subset \mathbb{R}^k$.

$I = [a_1, b_1] \times \cdots \times [a_k, b_k]$

To show E is compact, it suffices to show that I is compact.

Suppose, to get a contradiction, that there is an open cover $\{G_\alpha\}$ of I which has no finite subcover of I.

By dividing each interval $[a_j, b_j]$ into halves, $[a_j, \frac{a_j+b_j}{2}]$ and $[\frac{a_j+b_j}{2}, b_j]$, subdivide I into 2^k cubes of equal size.

At least one of them cannot be covered by any finite subcollection of $\{G_\alpha\}$; call it I_1.

Repeat this subdivision process to obtain a sequence $\{I_n\}$ of closed cubes such that

(1) $I \supset I_1 \supset I_2 \supset \cdots$,

(2) I_n is not covered by any finite subcollection of $\{G_\alpha\}$,

(3) $|\mathbf{x}-\mathbf{y}| \leqslant \frac{\delta}{2^n}$ for any $\mathbf{x}, \mathbf{y} \in I_n$, where $\delta := \left(\sum_{j=1}^{k} (b_j - a_j)^2\right)^{\frac{1}{2}}$.

Now, it suffices to prove that $\bigcap_{n=1}^{\infty} I_n$ is non-empty (for if $\mathbf{x}^* \in \bigcap_{n=1}^{\infty} I_n$, then $\mathbf{x}^* \in G_\alpha$ for some α, and in particular $B_r(\mathbf{x}^*) \subseteq G_\alpha$ for some $r>0$, but then $I_n \subset B_r(\mathbf{x}^*) \subseteq G_\alpha$ for any n big enough so that $\frac{\delta}{2^n} < r$, which is a contradiction to (2) above.)

In fact, $\bigcap_{n=1}^{\infty} I_n$ is just one point in this setup

5/ (proof continued)

Let's write $I_n = [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,k}, b_{n,k}] \subset \mathbb{R}^k$

Then, for each $1 \leq j \leq k$, $a_{1,j} \leq a_{2,j} \leq a_{3,j} \leq \cdots \leq b_{3,j} \leq b_{2,j} \leq b_{1,j}$.

Let $x_j^* = \sup\{a_{1,j}, a_{2,j}, a_{3,j}, \cdots\}$. ← using the l.u.b. property of $\mathbb{R}$!

Then clearly $a_{n,j} \leq x_j^* \leq b_{n,j}$ for all n.

x_j^* is an upper bound of $a_{n,j}$'s

each $b_{n,j}$ is an upper bound of $a_{m,j}$'s, and x_j^* is the least upper bound.

Therefore $\mathbf{x}^* := (x_1^*, \cdots, x_k^*) \in \bigcap_{n=1}^{\infty} I_n$ as desired. ■

<u>Cor</u> (Weierstrass) Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

proof) Being bounded, the subset in question is a subset of a closed k-dim'l cube, which is compact. The existence of a limit point then immediately follows from the previous thm ■